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EXACT ANALYTIC SOLUTIONS FOR STELLAR WIND BOW SHOCKS

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ABSTRACT

Stellar wind bow shocks have been seen in association with a wide variety of stellar objects, from pulsars to young stars. A new solution method is presented for bow shocks in the thin-shell limit, stressing the importance of the conserved momentum within the shell. This method leads to exact analytic solutions to the classical problem of Baranov, Krasnobaev, & Kulikovskii. Simple formulae are given for the shell's shape, mass column density, and velocity of shocked gas at all points in the shell. These solutions will facilitate detailed comparison between observed sources and bow shock models.

Subject headings: hydrodynamics — ISM: bubbles — shock waves — stars: mass loss

1. INTRODUCTION

Stellar wind bow shocks are cometary structures due to the supersonic passage of wind-blowing stars. They sweep up interstellar matter into thin, dense shells, which may be revealed by their postshock emission or by scattered light, and provide a means of studying winds that might otherwise go undetected. They have been observed associated with pulsars (see, e.g., Cordes, Romani, & Lundgren 1993), young B stars (Van Buren & McCray 1988), and even a cataclysmic variable (see Van Buren 1993 for a review).

The theory of momentum-supported bow shocks was first developed by Baranov, Krasnobaev, & Kulikovskii (1971, hereafter BKK), who were motivated by the problem of the interaction of the solar wind with the local interstellar medium. They considered the collision of an isotropic stellar wind with a uniform ambient medium, including the supersonic motion of the star with respect to that medium, and solved numerically for the shape of the bow shock.

In this Letter, I present a new formulation of the problem of momentum-supported bow shocks, which emphasizes the conserved internal momentum in the shell. This method allows one to obtain simple, exact solutions for all quantities in the numerical model of BKK: the shell's shape and the mass and velocity distributions within it. The shocked fluid in the shell has momentum that is precisely the vector sum of the momenta imparted to it by the wind and ambient medium, integrated over the surface of the shell, so there is no need to divide the problem into normal and tangential parts in curvilinear coordinates. Moreover, the tail of the bow obeys a simple, momentum-conserving snowplow model in two dimensions.

2. MATHEMATICAL FORMULATION

The stellar wind drives a shock into the ambient medium while the supersonic wind is abruptly decelerated, leading to two layers of shocked gas. These layers are assumed to mix, and postshock cooling is assumed to be so efficient that the dense shell has negligible thickness compared to the distance to the star. The star moves with constant velocity of magnitude V_* in a uniform medium of density ρ_a. The isotropic stellar wind has mass-loss rate $\dot{m}_w$ and constant speed V_w, yielding a cometary structure with the stellar velocity vector as symmetry axis. The flow is assumed to be hypersonic, so pressure forces are neglected. In this idealized model, the thin shell is fully described by three quantities: the shell's radius $R(\theta)$, mass surface density $\sigma(\theta)$, and the tangential speed $v_t(\theta)$ of shocked material flowing along the shell, where θ is the polar angle from the axis of symmetry, as seen by the star at the coordinate origin.

Let the z-axis be the axis of symmetry of the shell, with the stellar motion in the $\hat{\boldsymbol{e}}_z$-direction (toward the right in Fig. 1). In the frame of the star, the ambient medium appears as a uniform wind in the $-\hat{\boldsymbol{e}}_z$-direction. The stellar wind and the ambient medium collide head-on at $\theta = 0$, and the radius of this starting point of the shell is found by balancing the ram pressures of the wind and ambient medium, $\rho_w V_w^2 = \rho_a V_*^2$, which yields (BKK)

$$R_0 = \sqrt{\frac{\dot{m}_w V_w}{4\pi\rho_a V_*^2}}. \tag{1}$$

This standoff distance sets the length scale of the shell. The shape of the shell is a universal function, which is scaled according to equation (1) to accommodate all values of the four dimensional parameters ($\dot{m}_w$, V_w, ρ_a, V_w).

The fluxes of mass and momentum crossing an annulus of the shell, $2\pi\Phi_m(\theta)$ and $2\pi\Phi_t(\theta)$, are given respectively by

$$\Phi_m = (R\sin\theta)\sigma v_t, \quad \Phi_t = (R\sin\theta)\sigma v_t^2. \tag{2}$$

In steady state, the mass traversing a ring of the shell at polar angle θ from the standoff point is precisely that mass flux from the stellar wind intercepted by the solid angle of the forward part of the shell plus the contribution from the ambient medium striking the circular area of the projected cross section of the shell (BKK):

$$2\pi\Phi_m = \dot{m}_w \frac{\Omega}{4\pi} + \pi\varpi^2\rho_a V_*, \tag{3}$$

where $\Omega = 2\pi(1 - \cos\theta)$ is the shell solid angle from the axis to the annulus at θ, and the cylindrical radius ϖ is $R\sin\theta$. The remaining part of the calculation was solved by BKK by writing down ordinary differential equations for momentum conservation tangential to the shell and force balance normal to it and integrating these equations numerically. The resulting bow shock shape is shown in Figure 2, labeled "BKK."

Let us calculate the rate at which vector momentum is

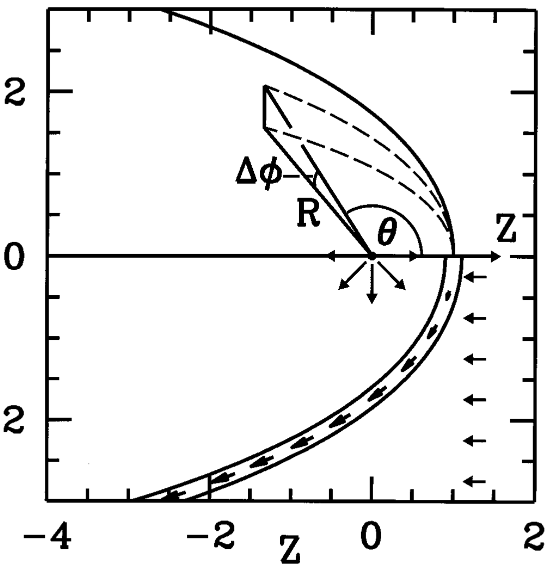

FIG. 1.—Thin-shell bow shock model. The bottom panel shows the wind, ambient, and tangential flows in the frame of the star while the top panel defines the spherical coordinate system with star at the origin.

imparted to the shell by the stellar wind. For this axisymmetric problem we may consider a wedge of small, constant width in the azimuthal angle $\Delta\phi$ about the symmetry axis (Fig. 1). The surface integral of the wind's vector momentum flux onto the shell does not depend on the detailed shape of the shell, because the coasting wind is momentum conserving. Thus, we may perform the integral over a spherical surface. Writing the

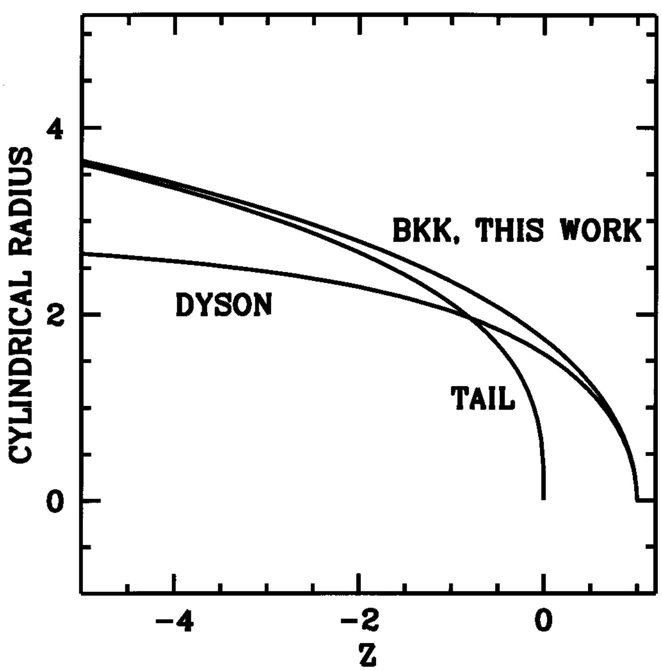

FIG. 2.—Bow shock shape of BKK and analytic solution (eq. [9]), compared to the approximate tail solution and the solution of Dyson (1975), which assumes that the shape is determined by normal ram-pressure balance.

unit vector $\hat{\boldsymbol{e}}_r$ in terms of its cylindrical polar components, we have

$$\boldsymbol{\Phi}_w \Delta\phi = \int_{\text{wedge}} \rho_a V_w \boldsymbol{V}_w \cdot \hat{\boldsymbol{e}}_n \, dA$$

$$= \frac{\dot{m}_w V_w}{4\pi} \Delta\phi \int_0^\theta (\hat{\boldsymbol{e}}_\varpi \sin\theta' + \hat{\boldsymbol{e}}_z \cos\theta') \sin\theta' \, d\theta'$$

$$= \frac{\dot{m}_w V_w}{8\pi} [(\theta - \sin\theta\cos\theta)\hat{\boldsymbol{e}}_\varpi + (\sin^2\theta)\hat{\boldsymbol{e}}_z]\Delta\phi. \tag{4}$$

The momentum deposited by the ambient medium is in the $-\hat{\boldsymbol{e}}_z$-direction and depends only upon the circular cross section:

$$\boldsymbol{\Phi}_a \Delta\phi = -\tfrac{1}{2}\varpi^2 \rho_a V_*^2 \hat{\boldsymbol{e}}_z \Delta\phi. \tag{5}$$

The total momentum flux onto the $\Delta\phi$ wedge of the shell is the sum of the wind and ambient contributions. To conserve momentum in steady state, the (tangential) momentum flux $\Phi_t \hat{\boldsymbol{e}}_t \Delta\phi$ traversing a $\Delta\phi$ azimuthal width of an annulus of the shell must equal the momentum flux $(\boldsymbol{\Phi}_w + \boldsymbol{\Phi}_a)\Delta\phi$ received by the shell surfaces between the standoff point and the annulus:

$$\boldsymbol{\Phi}_t = \frac{\dot{m}_w V_w}{8\pi} [(\theta - \sin\theta\cos\theta)\hat{\boldsymbol{e}}_\varpi + (\sin^2\theta)\hat{\boldsymbol{e}}_z] - \frac{\varpi^2}{2}\rho_a V_*^2 \hat{\boldsymbol{e}}_z, \tag{6}$$

where $\boldsymbol{\Phi}_t = \Phi_t \hat{\boldsymbol{e}}_t$ is the vector momentum flux in the shell and $\hat{\boldsymbol{e}}_t$ is a tangential unit vector at constant ϕ. The tangential momentum flux has magnitude

$$2\pi\Phi_t = \pi R_0^2 \rho_a V_*^2 \sqrt{(\theta - \sin\theta\cos\theta)^2 + (\tilde{\varpi}^2 - \sin^2\theta)^2}, \tag{7}$$

where a tilde indicates a length in units of R_0.

We now know the *vector* momentum flux at any point in the shell, so we know the direction of flow is parallel to this momentum flux. We conclude that the shell's shape is described by the differential equation

$$\frac{d\tilde{z}}{d\tilde{\varpi}} = \frac{v_z}{v_\varpi} = \frac{\Phi_{t,z}}{\Phi_{t,\varpi}} = \frac{-\tilde{\varpi}^2 + \sin^2\theta}{\theta - \sin\theta\cos\theta}. \tag{8}$$

It is instructive to examine the stellar and ambient momentum contributions separately (eqs. [4], [5]). Figure 3 shows the vectors $\boldsymbol{\Phi}_w$ and $\boldsymbol{\Phi}_a$ due to the wind and ambient momentum fluxes onto the shell, integrated in area between the standoff point and the angle of interest θ. The ambient medium contributes momentum purely in the $-\hat{\boldsymbol{e}}_z$-direction, while the wind contribution is not radial because the momentum flux onto the area of the head of the shell includes contributions from points where the wind velocity is more in the forward direction. The momentum leaving the shell must be the vector sum of these two contributions, which is precisely tangent to the shell at this point. The shell's shape is not determined by ram pressures forcing the radius to have a certain value at a given angle, nor a certain slope, but rather by the requirement that the fluid travel in the natural direction—that of the momentum deposited onto the shell, integrated over shell area.

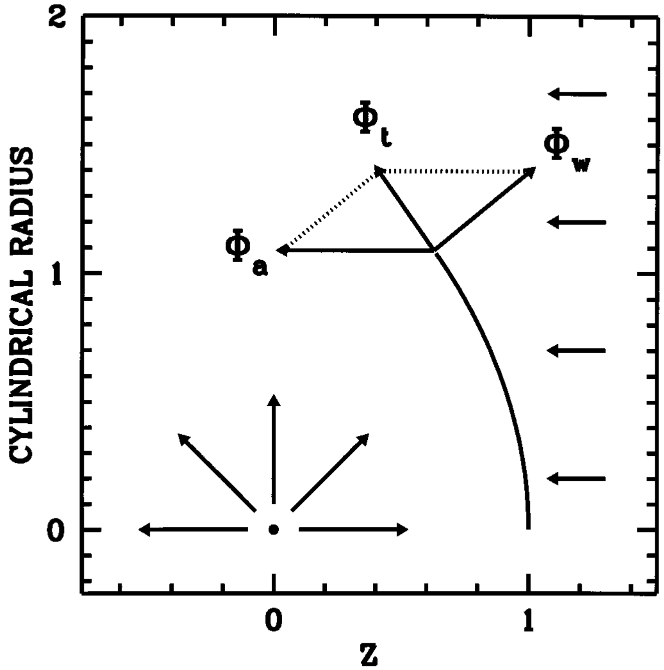

FIG. 3.—Vector momentum flux in the shell compared to the wind and ambient-medium integrated surface fluxes of momentum, demonstrating global vector momentum conservation.

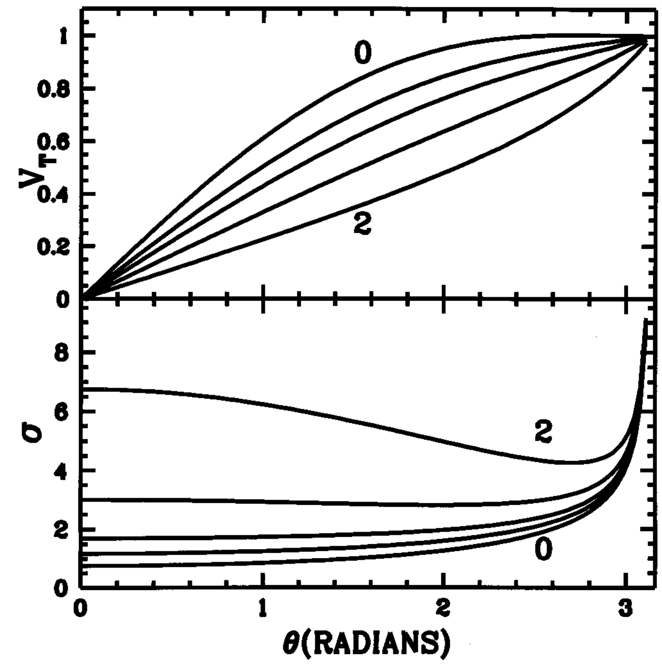

FIG. 4.—Internal shell velocity in units of V_* (*top*) and mass surface density in units of $R_0\rho_a$ (*bottom*) for several values of α. Curves for $\alpha = 0$ and 2 are labeled, and intermediate curves correspond to $\alpha = \frac{1}{4}, \frac{1}{2}$, and 1. The mass per unit area σ is defined for a line of sight that is normal to the shell. The normal direction to the shell may be obtained from eq. (6), using the relation $\hat{e}_n = \hat{e}_t \times \hat{e}_\phi$. If viewed from angle ϑ to the normal, the mass column σ must be modified by a projection factor sec ϑ so long as the path length of the line of sight through the thickness of the shell is short compared to the shell's curvature.

The transformation from $(\tilde{\varpi}, \tilde{z})$ to $(\tilde{r}, \theta)$ coordinates for the left-hand side of equation (8) may be written

$$\left(\cos\theta - \frac{d\tilde{z}}{d\tilde{\varpi}}\sin\theta\right)\tilde{r}' = \tilde{r}\left(\sin\theta + \frac{d\tilde{z}}{d\tilde{\varpi}}\cos\theta\right),$$

where the prime denotes differentiation by θ. Multiplying by $\tilde{r}$ $\sin^3\theta$, making the variable substitution $u = \tilde{r}^2 \sin^3\theta$, and using equation (8), one eventually obtains

$$u' = \frac{u[2\theta\sin\theta + \cot\theta(u + 3\theta\cos\theta - 3\sin\theta)]}{(u + \theta\cos\theta - \sin\theta)}.$$

We may now verify that $u = 3(\sin\theta - \theta\cos\theta)$ is the solution, giving an exact integral for the shell's shape:

$$R(\theta) = R_0 \csc\theta\sqrt{3(1 - \theta\cot\theta)}. \quad (9)$$

This formula for $R(\theta)$, together with the momentum flux $\Phi_t(\theta)$ of equation (7), is the solution of the equations of BKK with the desired initial conditions $R(0) = R_0$ and $R'(0) = 0$. With the previously known mass integral, we may now describe fully the remaining shell properties. Using equation (9), we have

$$\tilde{\varpi}^2 = 3(1 - \theta\cot\theta). \quad (10)$$

Referring to equation (2), the tangential velocity in the shell is $v_t = \Phi_t/\Phi_m$, which yields

$$v_t = V_* \frac{\sqrt{(\theta - \sin\theta\cos\theta)^2 + (\tilde{\varpi}^2 - \sin^2\theta)^2}}{2\alpha(1 - \cos\theta) + \tilde{\varpi}^2}, \quad (11)$$

where $\alpha \equiv V_*/V_w$ is a nondimensional parameter of the problem. The mass surface density is given by $\sigma = \Phi_m^2/(R\sin\theta)\Phi_t$, which yields

$$\sigma = R_0\rho_a \frac{[2\alpha(1 - \cos\theta) + \tilde{\varpi}^2]^2}{2\tilde{\varpi}\sqrt{(\theta - \sin\theta\cos\theta)^2 + (\tilde{\varpi}^2 - \sin^2\theta)^2}}. \quad (12)$$

The tangential velocity and mass surface density are plotted in Figure 4 for several values of α. For most systems, α is expected to be small, and the tangential velocity should prove a more sensitive way of obtaining α than the mass column.

The behavior of the analytic solutions near the standoff point is of interest for comparison with observed sources. Taylor series about $\theta = 0$, for the radius and momentum flux, yield $R(\theta) \approx R_0(1 + \theta^2/5 + 29\theta^4/1050)$ and $\Phi_t \approx \frac{1}{3}R_0^2\rho_a V_*^2\theta^3$ $(1 - \theta^2/50 + 479\theta^4/105{,}000)$. The leading-order correction for the radius was given by BKK. The approximate solutions of Dyson (1975) and Van Buren et al. (1990) for $R(\theta)$ both have Taylor series with lowest order correction $\theta^2/6$, with an error of ~17%, but only 10% in the radius of curvature. The tangential velocity v_t and mass surface density σ are $v_t \approx 2V_*\theta/3(1 + \alpha)$ and $\sigma \approx 3R_0\rho_a(1 + \alpha)^2/4$.

In the limit $\tilde{\varpi} \to \infty$, equations (7), (11), and (12) all have intuitively obvious asymptotic forms, where the mass and momentum are dominated by the ambient medium's contributions. We now derive the momentum-conserving snowplow model for this bow shock tail. Note from equation (4) that, in the frame of the star, the total wind momentum deposited in a $\Delta\phi$ wedge of the shell, integrated from $\theta = 0$ to $\theta = \pi$, is in the cylindrical direction $\hat{e}_\varpi$. We apply an impulse approximation, stating that a patch of the shell of angular width $\Delta\phi$ and length Δz receives a total momentum given by the crossing time to pass the star, $\Delta z/V_*$, times the stellar momentum-loss rate $\mathbf{\Phi}_w(\pi)\Delta\phi$. This cylindrical momentum $m_{\rm patch}V_\varpi = \dot{m}_w V_w$ $\Delta z\Delta\phi/8V_*$ is subsequently conserved as the shell annulus expands. Now the mass of the patch is dominated by the swept-up ambient medium for large ϖ, so we have $m_{\rm patch} = \varpi^2\rho_a\Delta z\Delta\phi/2$, and combining the expressions for the patch mass and momentum, we may integrate $V_\varpi = d\varpi/dt$ to obtain $\varpi^3 = 3\pi R_0^2 V_* t$, where we have used equation (1). The z-motion (eq. [11]) reduces to $z = -V_* t$, so we obtain the tail solution as $\tilde{z} = -\varpi^3/3\pi$, which is shown in Figure 2. This tail solution provides an excellent approximation for large θ and shows that, in the ambient frame (where $V_z \approx 0$), the tail is a momentum-conserving snowplow in two dimensions, with $\hat{\boldsymbol{e}}_z$ serving as the "timelike" direction in steady state.

3. CONCLUSIONS

These results should have wide applicability in future observational studies of bow shocks and facilitate further theoretical computations of mixing in the shocked layer (Raga, Cabrit, & Cantó 1995), stability of the shell (Dgani, Van Buren, & Noriega-Crespo 1996), and the expected emission. The physical parameters of real sources can now be obtained more easily, such as the deprojection of the shape to obtain the true vector velocity of a source given the apparent shape of the bow (Mac Low et al. 1991).

Because the bow shock, in dimensional units, depends upon both the stellar wind's and ambient medium's properties, stellar wind bow shocks may be very useful probes of both. Spatially resolved observations can hope to recover the orientation of the bow and, hence, convert the photospheric radial velocity to a true spatial velocity. With spatially resolved velocity information, we can recover the mass and momentum flux functions and thus the stellar mass-loss and momentum-loss rates. Given the distance, the angular size determines the true size, and we obtain the local ambient density from equation (1). It is the author's hope that this simple solution will motivate observers to attempt to derive all the physical parameters of the model from detailed observations of individual objects.

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REFERENCES

Baranov, V. B., Krasnobaev, K. V., & Kulikovskii, A. G. 1971, Soviet Phys.–Dokl., 15, 791
Cordes, J. M., Romani, R. W., & Lundgren, S. C. 1993, Nature, 362, 133
Dgani, R., Van Buren, D., & Noriega-Crespo, A. 1996, ApJ, in press
Dyson, J. 1975, Ap&SS, 35, 299
Mac Low, M.-M., Van Buren, D., Wood, D. O. S., & Churchwell, E. 1991, ApJ, 369, 395
Raga, A. C., Cabrit, S., & Cantó, J. 1995, MNRAS, 273, 422
Van Buren, D. 1993, in ASP Conf. Proc. 35, Massive Stars: Their Lives in the Interstellar Medium, ed. J. P. Cassinelli & E. B. Churchwell (San Francisco: ASP), 315
Van Buren, D., Mac Low, M.-M., Wood, D. O. S., & Churchwell, E. 1990, ApJ, 353, 570
Van Buren, D., & McCray, R. 1988, ApJ, 329, L93